

PHASE ORDERING IN CHAOTIC MAP LATTICES WITH ADDITIVE NOISE

Leonardo Angelini, Mario Pellicoro, and Sebastiano Stramaglia

Dipartimento Interateneo di Fisica

Istituto Nazionale di Fisica Nucleare, Sezione di Bari

via Amendola 173, 70126 Bari, Italy

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Abstract

We present some result about phase separation in coupled map lattices with additive noise. We show that additive noise acts as an ordering agent in this class of systems. In particular, in the weak coupling region, a suitable quantity of noise leads to complete ordering. Extrapolating our results at small coupling, we deduce that this phenomenon could take place also in the limit of zero coupling.

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I. INTRODUCTION

Non trivial collective behavior (NTCB) is an interesting feature peculiar to extensively chaotic dynamical systems, where the temporal evolution of spatially averaged quantities reveals the onset of long-range order in spite of the local disorder [1] [2] [3]. A simple way to observe NTCB is to study models of spatially extended chaotic systems such as coupled map lattices (CMLs) that consist of chaotic maps locally coupled diffusively with some coupling strength g [3]. In these systems one observes multistability that is the remainder, for small couplings, of the completely uncoupled case [4]. For large enough couplings NTCB is observed, corresponding to a *macroscopic* attractor, well-defined in the infinite-size limit and reached for almost every initial condition.

In a recent paper [5] phase separation mechanisms have been investigated in a coupled map lattice model where the one-body probability distribution functions of local (continuous) variables has two disjoint supports. By introducing Ising spin variables, the phase ordering process following uncorrelated initial conditions was numerically studied and complete phase ordering was observed for large coupling values. Both the persistence probability $p(t)$ (i.e. the proportion of spins that has not changed sign up to time t) and the characteristic length of domains $R(t)$ (evaluated as the width at midheight of the two-point correlation function) showed scaling behavior and the scaling exponents z and θ (defined by the scaling laws $R(t) \sim t^z$ and $p(t) \sim t^{-\theta}$) were found to vary continuously with parameters, at odds with traditional models. The study of the phase ordering properties also allowed to determine the limit value g_c beyond which multistability disappears and NTCB is observed [5]. Indeed the following relations were found to hold: $\theta \sim (g - g_c)^w$ and $z \sim (g - g_c)^w$, and were used to determine g_c . The ratio θ/z was found to be close to 0.40, the ratio known for the time dependent Ginzburg-Landau equation.

Subsequently dynamical scaling was studied in a lattice model of chaotic maps where the corresponding Ising spin model conserves the order parameter [6]. This model is equivalent to a conserved Ising model with couplings that fluctuate over the same time scale as spin

moves, in contact with a thermal bath at temperature T . The scaling exponents θ and z were found to vary with temperature. In particular the growth exponent z was observed to increase with temperature; it follows that thermal noise speeds up the phase ordering process in this class of models. At high temperatures z assumes the value $1/3$, corresponding to the universality class of a Langevin equation known as model B [7], that describes the standard conserved Ising model (when bulk diffusion dominates over surface diffusion [8]).

The role of noise as an ordering agent has been broadly studied in recent years in the context of both temporal and spatiotemporal dynamics. In the temporal case, that was first considered, external fluctuations were found to produce and control transitions (known as *noise-induced transitions*) from monostable to bistable stationary distributions in a large variety of physical, chemical and biological systems [9]. As far as spatiotemporal systems are concerned, the combined effects of the spatial coupling and noise may produce an ergodicity breaking of a bistable state, leading to phase transitions between spatially homogeneous and heterogeneous phases. Results obtained in this field include critical-point shifts in standard models of phase transitions [10], pure *noise-induced phase transitions* [11], stabilization of propagating fronts [12], and noise-driven structures in pattern-formation processes [13]. In all these cases, the qualitative (and somewhat counterintuitive) effect of noise is to enlarge the domain of existence of the ordered phase in the parameter space.

It is the purpose of this paper to analyse the role of additive noise in the phase separation of multiphase coupled map lattices [14]. It will be shown that external noise can induce complete phase ordering for coupling values not leading to phase separation in the absence of the noise term. Furthermore this dynamical transition is reentrant: phase separation appears at a critical value of the noise intensity but disappears again at one higher value of the noise strength.

The paper is organized as follows. In the next section the coupled map lattice model here considered is introduced. In section 3 we present our numerical results. Section 4 summarizes our conclusions.

II. THE MODEL

Let us consider a two-dimensional square lattice of coupled identical maps f acting on real variables x_i , whose evolution is governed by the difference equation:

$$x_i(t+1) = (1-4g)f[x_i(t)] + g \sum_{\langle ij \rangle} f[x_j(t)] + \xi_i(t). \quad (1)$$

Here the sum is over the nearest neighbors of site i , ξ_i is a random number uniformly distributed in $[-\sigma/2, \sigma/2]$, g is the coupling strength and periodic boundary conditions are assumed. We have chosen the following map:

$$f(x) = \begin{cases} -\frac{\mu}{3} \exp[\alpha(x + \frac{1}{3})] & \text{if } x \in [-\infty, -\frac{1}{3}] \\ \mu x & \text{if } x \in [-\frac{1}{3}, \frac{1}{3}] \\ \frac{\mu}{3} \exp[\alpha(\frac{1}{3} - x)] & \text{if } x \in [\frac{1}{3}, +\infty] \end{cases} \quad (2)$$

that is defined for every x in the real axis (see Fig. 1). The map here considered is a modified version of the map used in [5]; the modification is motivated by the fact that, due to the term noise ξ_i , variables $x_i(t)$ are not constrained to take value in $[-1, 1]$. Choosing $\mu = 1.9$ and $\alpha = 6$, f has two symmetric chaotic attractors, one with $x > 0$ and the other with $x < 0$. In Fig. 2 we show the invariant distribution of the attractor with positive x 's: it is composed of smooth pieces. The Lyapunov exponent of the map was evaluated 0.558.

To study the phase ordering process, uncorrelated initial conditions were generated as follows: one half of the sites were chosen at random and the corresponding values of x were assigned according to the invariant distribution of the chaotic attractor with $x > 0$, while the other sites were similarly assigned values with $x < 0$. We associated an Ising spin configuration $s_i(t) = \text{sgn}[x_i(t)]$ with each configuration of the x variable. Large lattices (up to 1000×1000) with periodic boundary conditions were used; the persistence $p(t)$ was measured as the proportion of sites that has not changed s the initial value. The average domain size $R(t)$ was measured by the relation $C[R(t), t] = 1/2$, where $C(r, t) = \langle s_{i+r}(t)s_i(t) \rangle$ is the two point correlation function of the spin variables. Both $p(t)$ and $R(t)$ were averaged over many (up to thirty) different samples of initial conditions.

III. RESULTS

Fixing $\sigma = 0$, that is considering the noise-free case, we performed the analysis suggested in [5]. For various values of g we measured the characteristic length R and the persistence p as functions of time; both these quantities saturate for small couplings and show scaling behaviour for large g values. The associated exponents, respectively z and θ , were continuous functions of g well described by the fitting ansatz [5]:

$$z \sim (g - g_c)^w, \quad \theta \sim (g - g_c)^w. \quad (3)$$

The estimated values of g_c and w were $g_c = 0.1652$ and $w = 0.2260$ while fitting the exponent z , and $g_c = 0.1654$, $w = 0.2105$ for the exponent θ . The ratio θ/z was approximately independent of g and equal to 0.3767. Furthermore, we observed that the same fitting ansatz can be used to fit our data for nonvanishing and small noise strength σ . For example, in Fig. 3(a) and 3(b) we show respectively the fit of z and θ versus g , while keeping σ fixed and equal to 0.1. As one can see, data are well fitted by the scaling forms (3), and the estimated values are $g_c = 0.1628$, $w = 0.2197$ for the z exponent, and $g_c = 0.1632$, $w = 0.2024$ for the θ exponent [15]. The ratio θ/z was estimated at 0.3838. We remark that our estimate of the critical coupling g_c , when non-vanishing and small noise is present, is smaller than the noise-free critical value. This fact clearly shows that a proper amount of noise favours the phase separation process of the system.

Let us now consider the region $g < g_c(\sigma = 0) = 0.165$. Here in the noise-free case the system evolves towards blocked configurations and no phase separation takes place. We checked, however, that this asymptotic regime was attained after very long evolution times: the system spented a lot of time in metastable states, so that the evolution curve for R and p displayed typical stairs structure. This structure (the times marking the steps of the curve) was very robust, in the sense that:

- it resisted to a change of the initial conditions (chosen following the particular prescription of section II),

- it did not depend on lattice dimension,
- a little noise (low σ) did not destroy it.

Nevertheless, when growing the amount of noise, the life time of these metastable states became shorter and shorter, till they definitely disappeared for σ greater than a critical value $\sigma_c(g)$. For $\sigma > \sigma_c(g)$ we got again power laws for $R(t)$ and $p(t)$, showing that the system separates for large times. This behaviour is shown in Fig. 4.

We estimated the critical value σ_c by fitting our data with the ansatz $z \sim (\sigma - \sigma_c)^w$. In Fig. 5 we show our data corresponding to $g = 0.16$: we evaluated $\sigma_c = 0.1094$ and $w = 0.3152$. As in the case of the choice 3, we have no theoretical argument to support the choice of the fitting ansatz, but the fact that it works on a large interval of g letting us to give a precise measurement of σ_c . We were able to measure in such a way σ_c for g greater than 0.025; at smaller values of g the dynamics became very slow and we were not able to numerically extract the exponent z .

As σ was increased, we found a transition at another critical value of the noise strength showing that the system does not separate beyond this critical σ . As an example in Fig. 6 we show the exponent z versus σ for g fixed and equal to 0.17. The transition seems to be discontinuous.

We repeated this analysis for several values of g . Interpolating the above described data for the critical noise strength, we built the phase diagram for the model shown in Fig. 7. The system separates in the shaded area, that is it tends asymptotically to complete phase ordering. Points in the white area correspond to an asymptotic regime of the system where clusters of the two phases are dynamical but their mean size remains constant; only for $\sigma = 0$ one has blocked configurations with clusters fixed in time. Our data concern g greater than 0.025, however we extrapolated the two critical curves towards $g = 0$. We observe, interestingly, that the extrapolation of the two curves seem to meet at $g = 0$; further investigation is needed to clarify the behavior of the noisy system close to $g = 0$.

IV. CONCLUSIONS

The phase ordering properties of multiphase chaotic map lattices have recently attracted interest since they differ from those of traditional models. In this paper we have shown that additive noise acts as an ordering agent in this class of systems, i.e. for a suitable amount of noise the system may order even for values of the coupling strength for which no separation is observed in the absence of the noise-term. A simple explanation for this behavior is as follows. Small values of the spatial coupling lead, in the noise-free case, to spatially blocked configurations where interfaces between clusters of each phase are strictly pinned. A proper amount of noise makes the system cross these barriers thus leading to complete ordering. We have numerically constructed a phase diagram describing this behavior. As we said, a similar effect was observed in chaotic map lattices evolving with conserved dynamics, where we found that the growth exponent increases with temperature [6]; in the present case the additive noise plays the role of the thermal noise.

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Figure Captions

Figure 1: The map $f(x)$ defined in (2).

Figure 2: Invariant probability distribution for the positive attractor of $f(x)$.

Figure 3: The estimated scaling exponents at fixed noise $\sigma = 0.1$: a) the dependence of the growth exponent z from g in linear and log-log plot, b) the dependence of the persistence exponent θ from g in linear and log-log scale. Solid lines are best fits leading to the determination of g_c and w through the use of (3).

Figure 4: The effect of additive noise on the time evolution of the domain size $R(t)$ at $g = 0.05$. The three curve are relative to $\sigma = 0, \sigma = 0.06, \sigma = 0.24$.

Figure 5: The estimated growth exponent z versus σ at fixed coupling $g = 0.16$ in linear and log-log scale. Also shown is the best fit with the function $z \sim (\sigma - \sigma_c)^w$.

Figure 6: The estimated growth exponent z versus σ at fixed coupling $g = 0.17$. z goes abruptly to zero at $\sigma = 1.2$ showing that the system does not separate beyond this threshold.

Figure 7: The phase diagram in the plane $\sigma - g$. The shaded area represents the parameter region in which the system separates asymptotically.

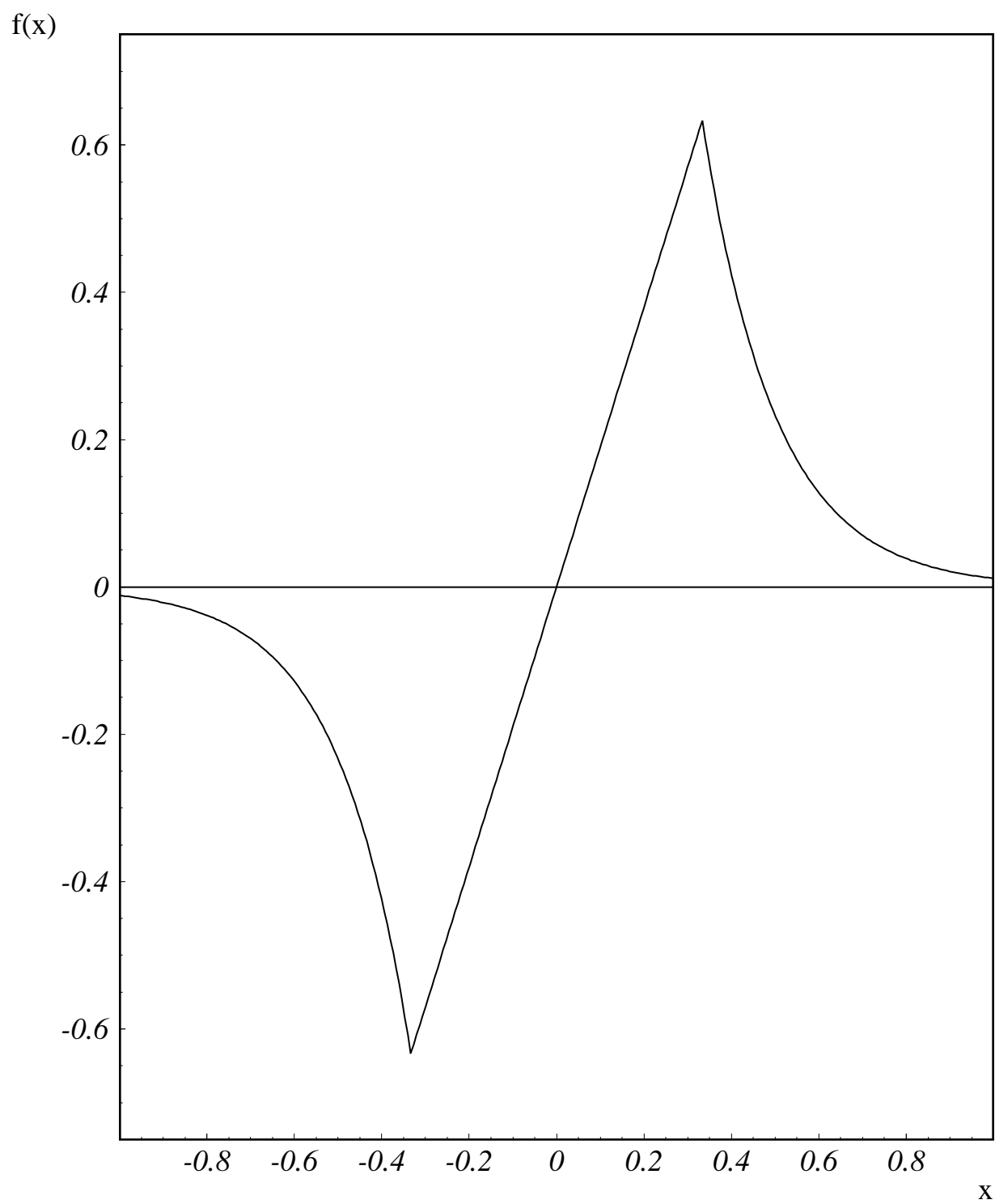


fig. 1

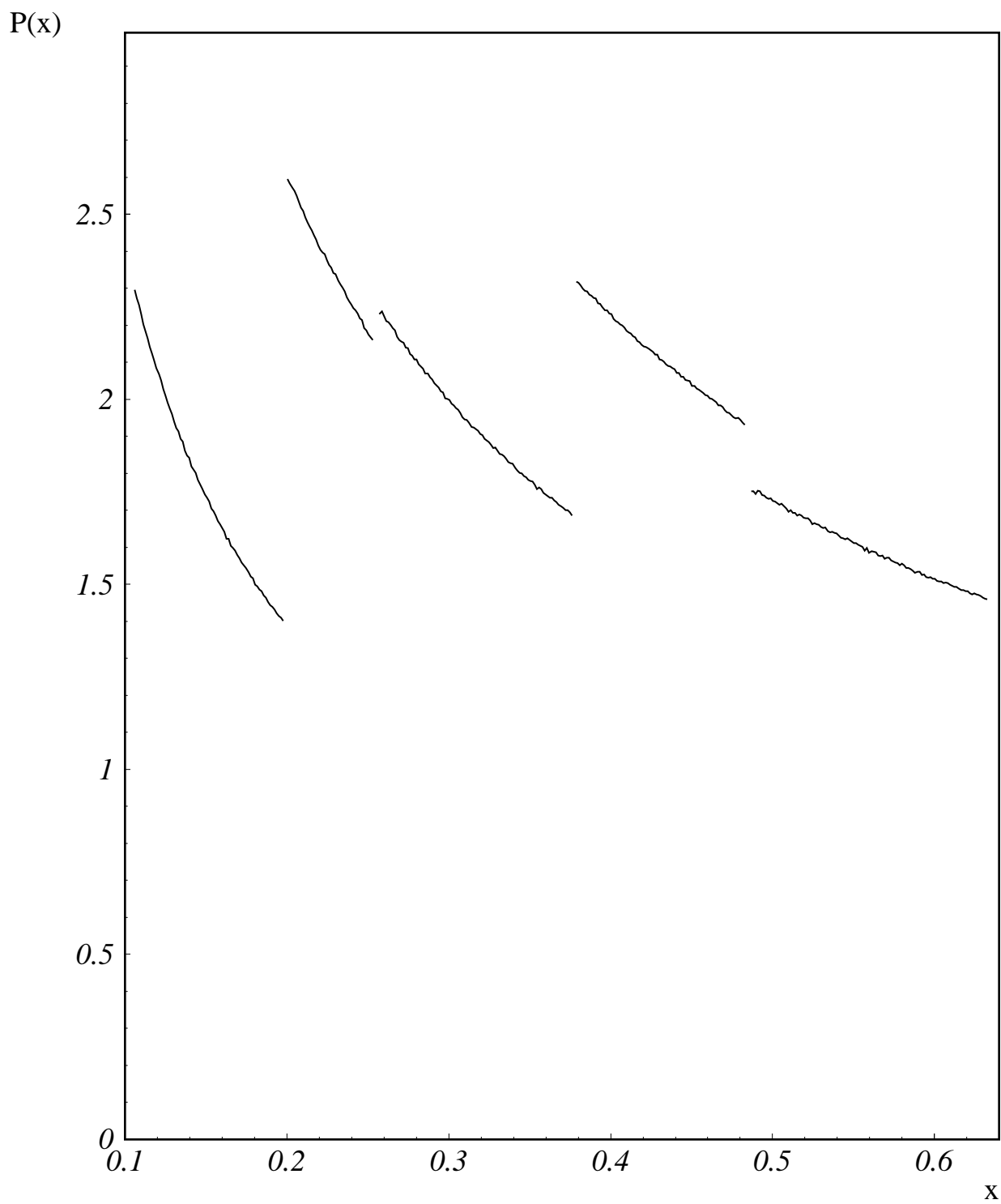
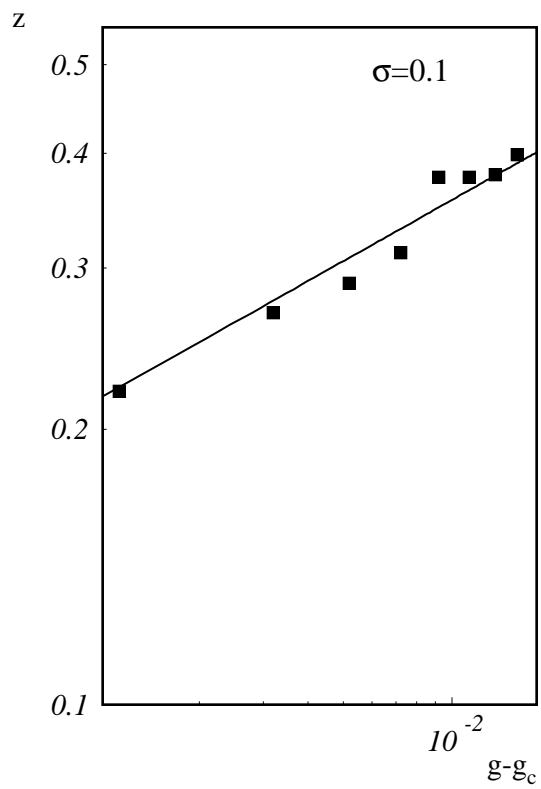
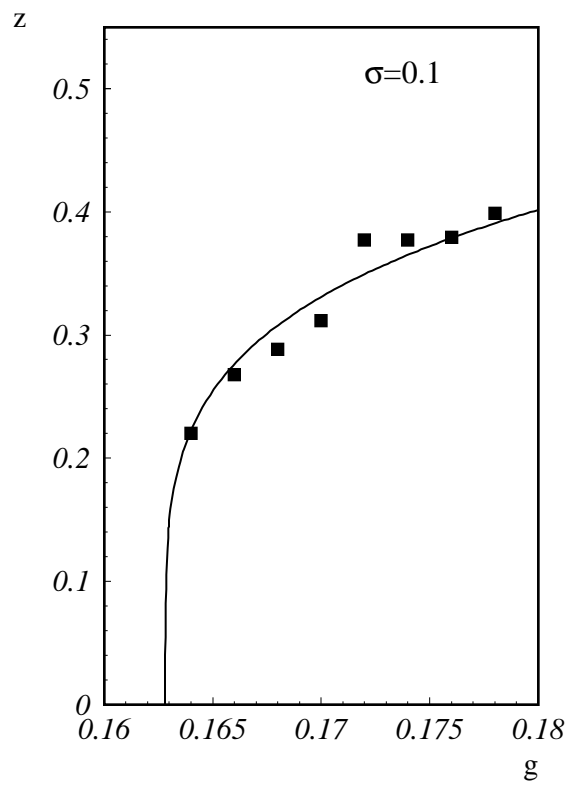
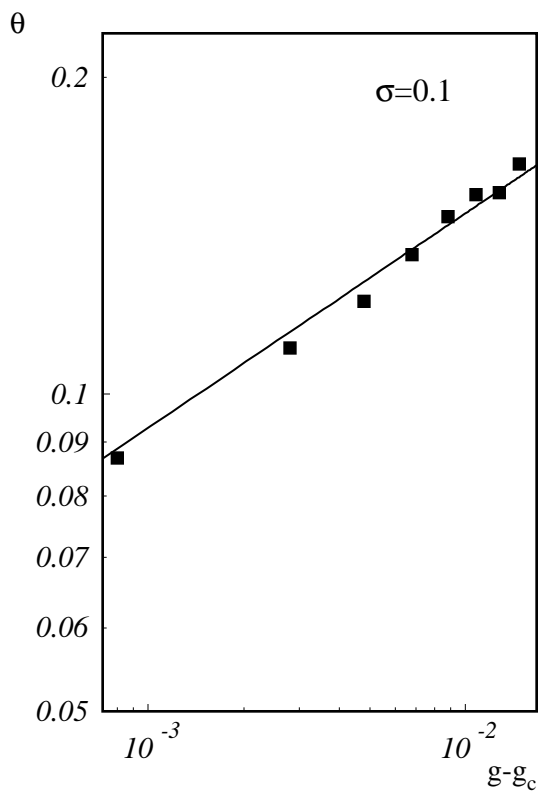
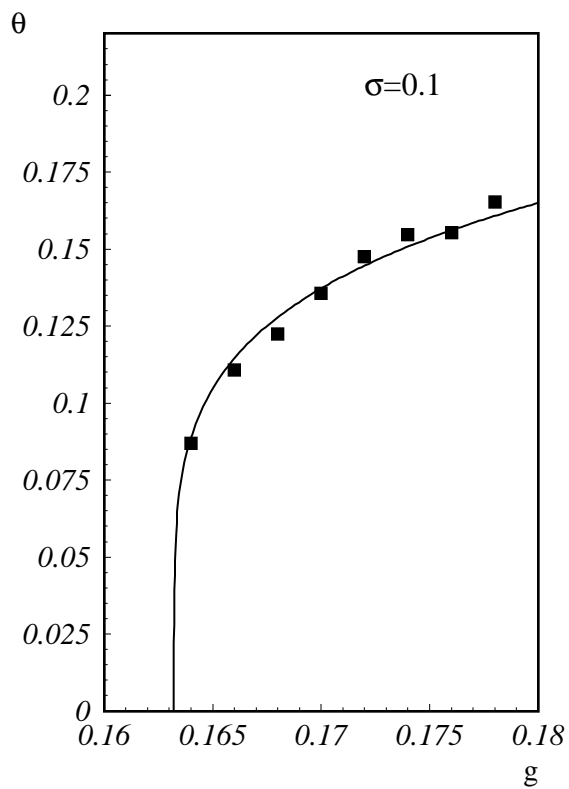


fig. 2



(a)



(b)
fig. 3

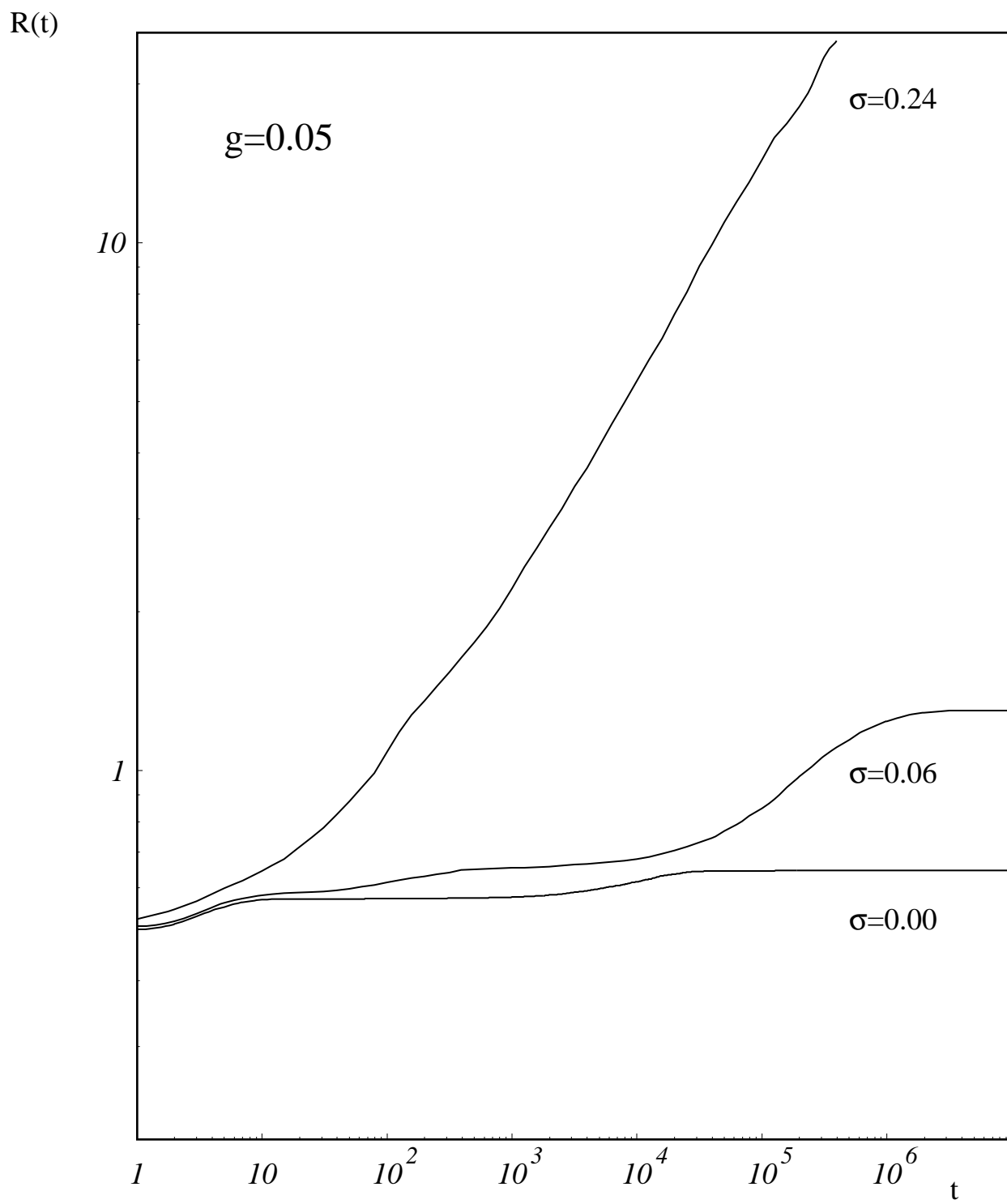


fig. 4

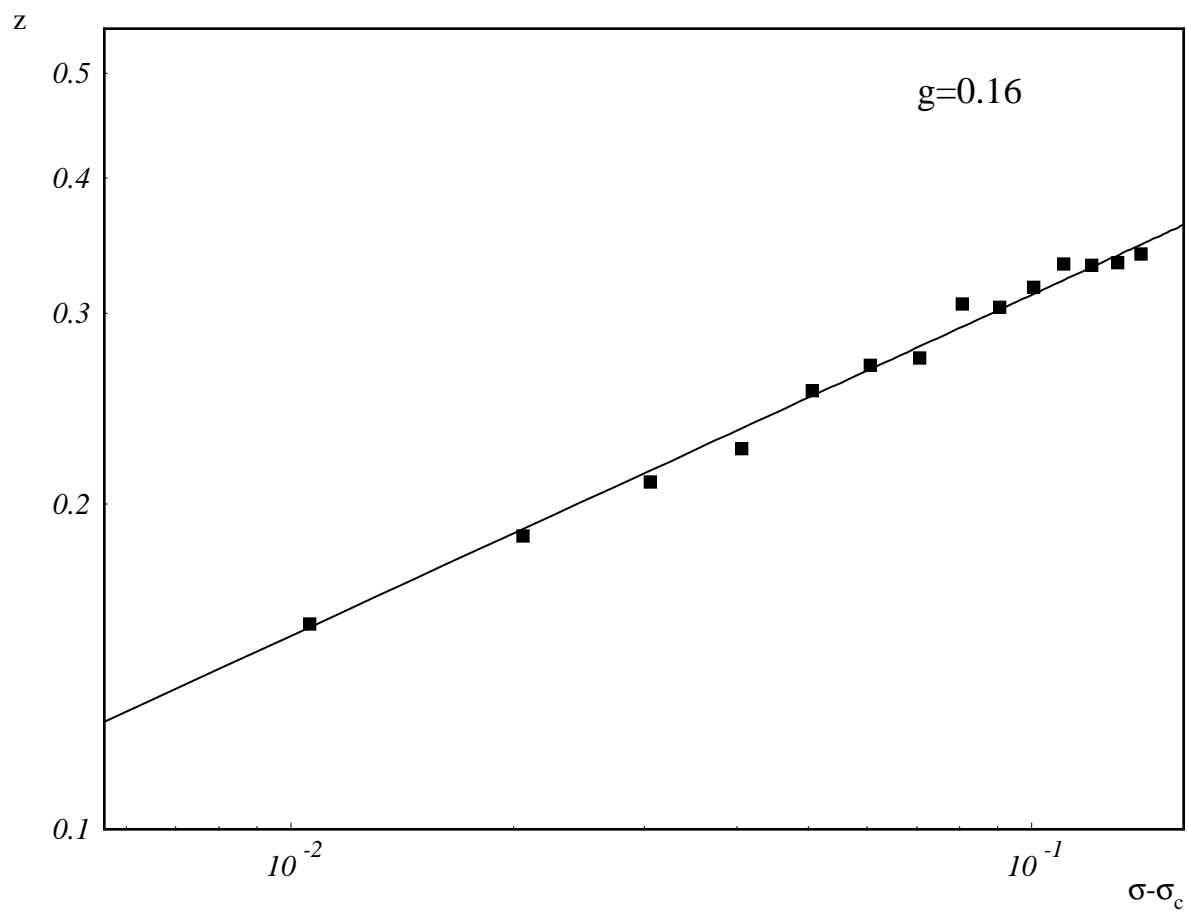
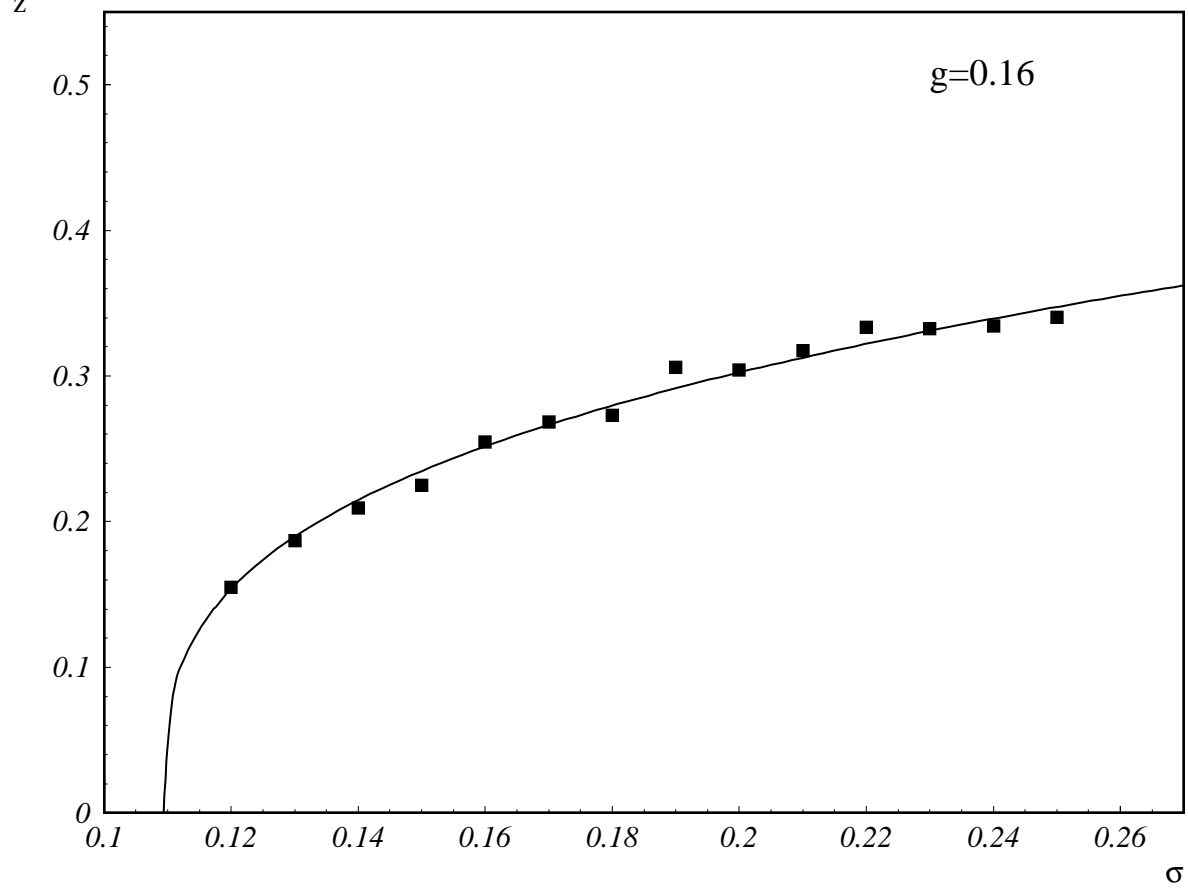


fig. 5

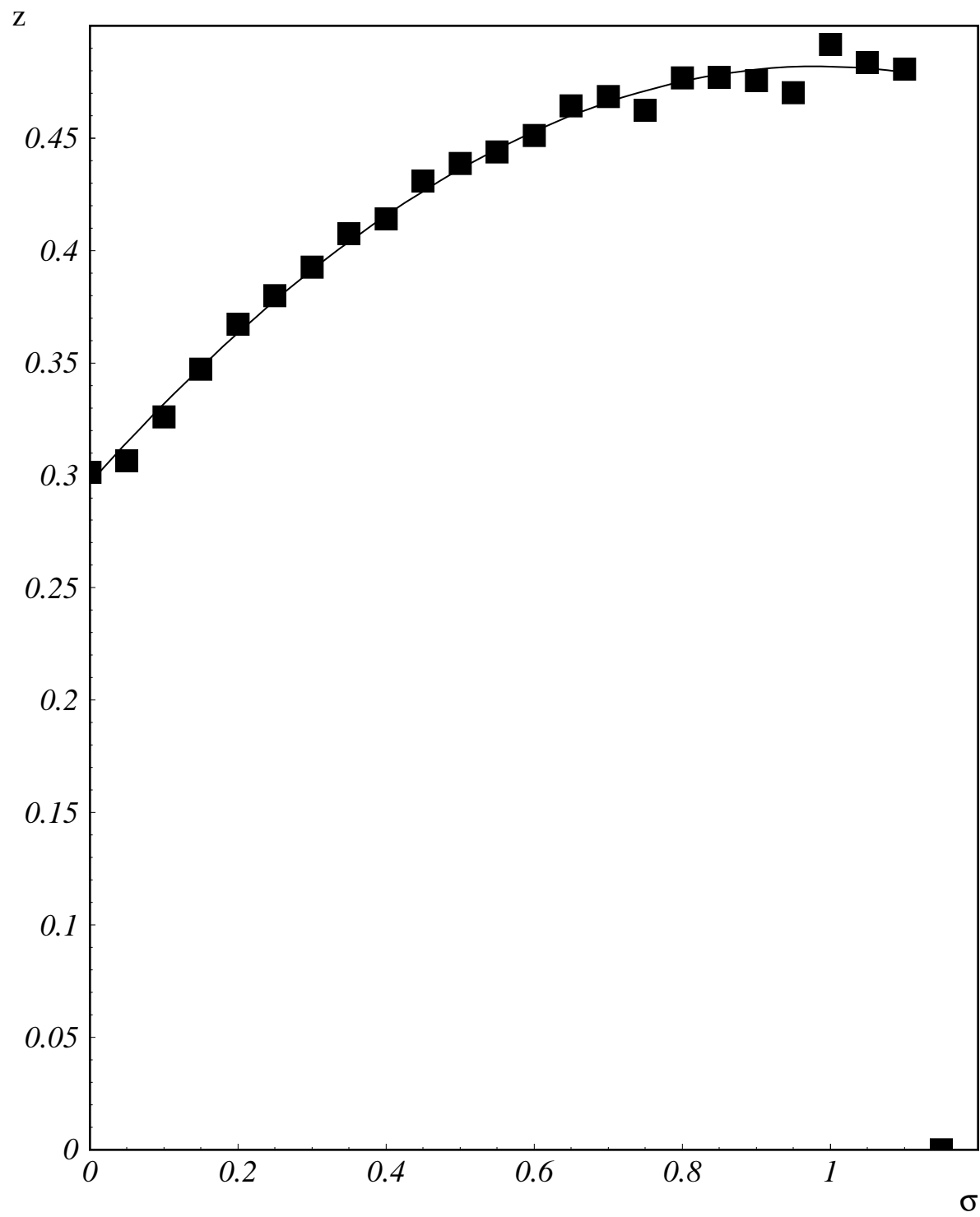


fig. 6

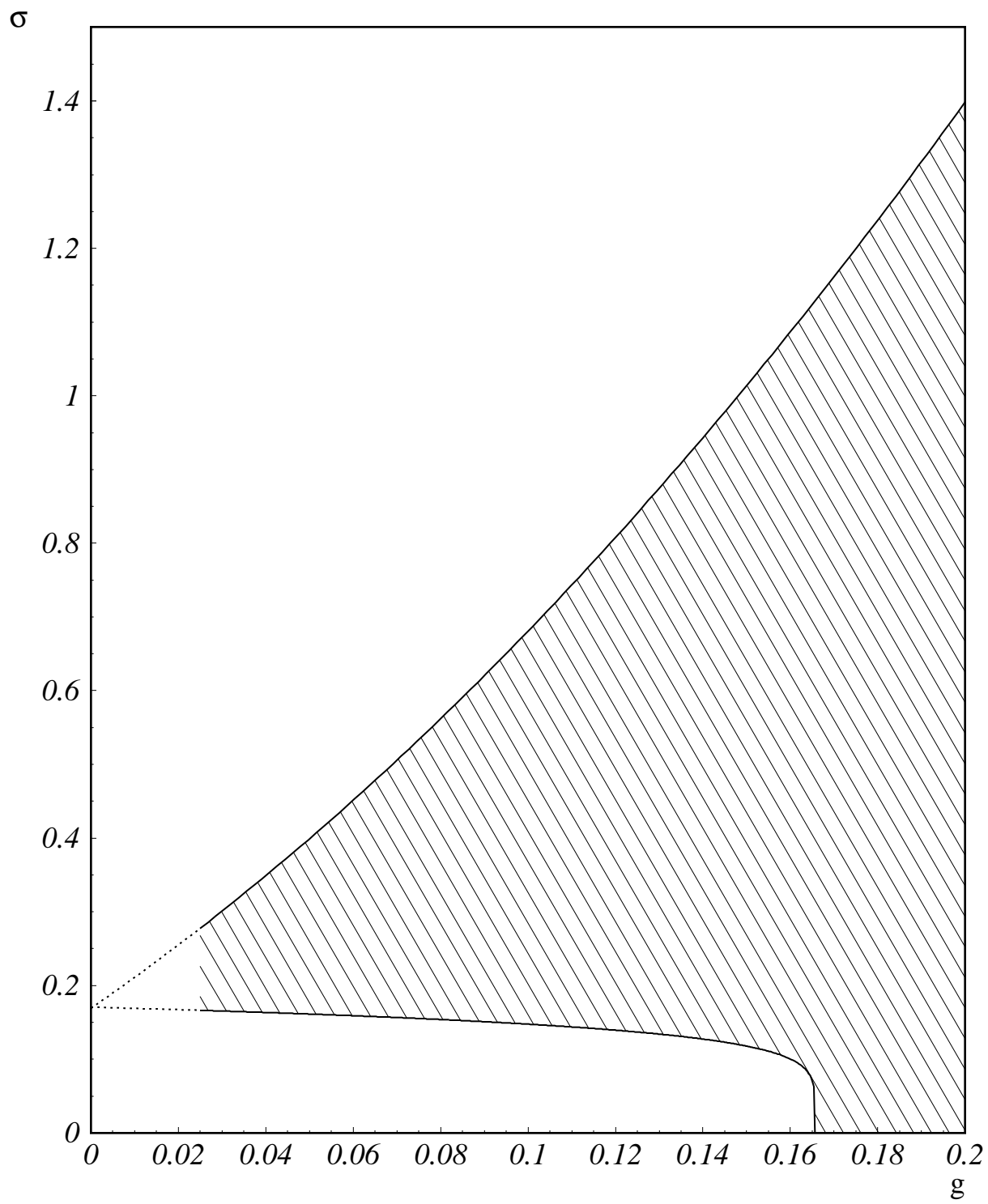


fig. 7